PULSE WAVE PROPAGATION IN ELASTIC TUBES HAVING LONGITUDINAL CHANGES IN AREA AND STIFFNESS

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ABSTRACT The behavior of both step waves and sinusoidal waves in fluid-filled elastic vessels whose area and distensibility vary with distance is explored theoretically. It is shown that the behavior of these waves may be explained, to a large extent, by considering the effect of the continuous stream of infinitesimal reflections that is set up whenever any wave travels in a region of vessel where the local impedance, (that is, the ratio of elastic wavespeed to tube area) is not constant. It is found that in such vessels the behavior of sinusoidal waves over distances which are a fraction of a wavelength can be quite different from their average behavior over several wavelengths. Both behaviors are described analytically. The results are applied to the mammalian circulatory system, one of the most interesting results being that a longitudinal variation in the pressure and velocity amplitudes which has a wavelength roughly one-half that of standing waves is predicted. The treatment is essentially a linearized quasi-one-dimensional one, the major assumptions being that the fluid is inviscid, the mean flow is zero, and the vessel is perfectly elastic and constrained from motion in the longitudinal direction. As in the physiological situation, the ratio of fluid velocity to pulse propagation speed is assumed small. For comparison with the analytical results, the linearized equations are also solved numerically by computer.

INTRODUCTION

The earliest attempts at analyzing blood flow in the major arteries (7, 18), which were quite successful (6) in relating the instantaneous pressure gradient to the instantaneous blood flow, were based on a model of an infinitely long, lengthwise invariant elastic tube. They predict, however, that owing to viscosity both the pressure pulse and velocity pulse should continually decrease in amplitude as they propagate from the heart towards the periphery, whereas in fact, in the large and medium size arteries the pressure pulse actually increases in amplitude towards the periphery. It is reasonable to suppose that the differences between the observations and the elementary theory arise from the lengthwise variations in cross-sectional area, wall-thickness, and stiffness of the vessels.

It has recently been discovered that the aorta increases in stiffness away from the heart, and that its branches have a still greater stiffness (11 and others). It is also well known that the cross-sectional area of the aorta decreases at least fivefold along its length. Several recent analyses have substantiated the view that these variations are important to the development of the pressure pulse contour (1, 2, 12, 15, 19). However, little has been reported concerning the relative magnitude of the changes in pulse contour as caused by changes in stiffness, changes in area, and reflections from junctions. Nor has it been made completely clear precisely what the important variables are.

Previous Work

The major calculational methods applied to pulsatile blood flow in a nonuniform tube have been, (*a*) solution of the nonlinear equations by means of the method of characteristics, (*b*) simulation of the arterial system by means of an electrical analogue, and (*c*) solution of the linearized equations.

The method of characteristics was first applied to blood flow problems by Streeter (15). This method, while potentially more accurate if significant nonlinearities are present, has the disadvantage that only numerical results, not broad analytical conclusions, are achievable. Barnard, Hunt, Timlake, and Varley (1), using the method of characteristics, have confirmed that where the area of a vessel decreases or the stiffness increases along the tube, the pressure pulse increases in amplitude while the velocity pulse decays. They also suggest that reflections set up in the nonuniform part of the tube play an important role.

Attempts to build an analogue of the arterial system with lumped electrical elements are characterized by the work of Noordergraaf and his associates (8). This method agains suffers from the fact that only numerical results are obtained.

Among the linearized solutions, Skalak and Stathis (12) treat the case where the area but not wall elasticity varies with distance. They also require that the ratio of wall-thickness to radius, as well as Poisson's ratio, remain constant along the tube. They represent the outflow from the aorta as a continuous flow through a porous wall. It is not clear that this retains the effect of gross reflections caused by the abrupt changes in flow which occur at junctions. Evans (2) gives the exact analytic solutions for a tube with an exponential change in area but no change in elastic wavespeed.

The Present Approach

In the theoretical treatment developed here, the tube properties are allowed to change by large amounts in the lengthwise direction. Despite the large longitudinal variation, the fluid mechanical equations of motion may be linearized if the amplitude of unsteady motion (as measured by the ratio of unsteady fluid velocity to the wave propagation speed (v/c)), is small. This approach is still valid for larger values of v/c, ($v/c \sim 0.1$ or 0.2), so long as the nonlinear behavior of the system as a whole is inherently small.

The effect of nonlinearities in the mammalian cardiovascular system has previously been discussed by McDonald (5). He concludes that both the effects of nonlinearities in the behavior of the aortic wall as well as nonlinearities due to the radial inertia of the fluid are small. Wylie (19) and Olsen and Shapiro (9), using the method of characteristics, also conclude the same. Thus it would appear that a linearized approach, such as the one presented here, could give considerable insight into the problem of pulse wave propagation as it exists in the mammalian cardiovascular system.

THE PROBLEM AND EQUATIONS

In the case of zero net flow and zero viscosity, the linearized equations, expressed in terms of pressure and volume flow, are identical in form with those of the non-uniform transmission line. With transmission lines, the important problems are those where the line is many wavelengths long and the line properties do not change much over one wavelength. Solutions to this problem have been thoroughly explored (4, 13, 16, 17).

The problem of interest to the mammalian physiologist is the case where the transmission line (artery) is a fraction of or only a few wavelengths long. In addition, certain of the tube (artery) properties may change by perhaps a factor of 10 or more over the entire length of the tube. This is the case we have considered. The assumptions made in the analysis are that the unsteady (oscillatory) fluid velocities are small with respect to the pulse propagation velocities or alternatively, that the nonlinear effects are small, and furthermore, that the flow may be treated as quasi-one-dimensional. The latter is a good approximation when the first assumption is valid and when, in addition, the fractional rate of change of rest area with distance is small and the wavelength of the disturbance in area is large compared with the diameter. It has also been assumed that the tube is perfectly elastic, although not necessarily linear in its elastic properties and that it is constrained from motion in the longitudinal direction. Under these conditions the linearized equations of motion for an inviscid fluid in an elastic tube whose area and extensibility vary with distance are known to be for the case of zero mean flow

$$\partial P_u/\partial x + \rho \partial V_u/\partial t = 0, \quad (1)$$

$$\partial(A_0 V_u)/\partial x + \partial A_u/\partial t = 0. \quad (2)$$

$P_u(x, t)$, $A_u(x, t)$, and $V_u(x, t)$ are the unsteady (oscillatory) components of the instantaneous internal pressure, area, and the average cross-sectional velocity in the x-direction and $p_0(x)$, $A_0(x)$, and $v_0(x)$ are the steady-state conditions about which the oscillatory perturbations take place. In the inviscid, zero mean flow case, p_0 is constant, v_0 equals zero, and A_0 is the internal rest area of the tube.

In a longitudinally constained, perfectly elastic tube where wall bending moments and radial inertia are unimportant, the instantaneous area at a point, x, is a function of only the instantaneous pressure at that point, and vice versa, so that a third basic relationship exists, namely

$$p(x, t) \equiv p_0 + P_u(x, t) = p(A, x). \tag{3}$$

The x variable in the last functional expression takes into account the variation along the tube of all the variables that influence the local pressure-area relationship.

BEHAVIOR OF A WAVE FRONT IN A TUBE WITH LENGTHWISE VARYING PROPERTIES

Consider first the behavior of a step wave, P_{in} incident upon a junction between two lengthwise invariant, albeit dissimilar tubes as in Fig. 1. A reflected wave, P_{re} and a

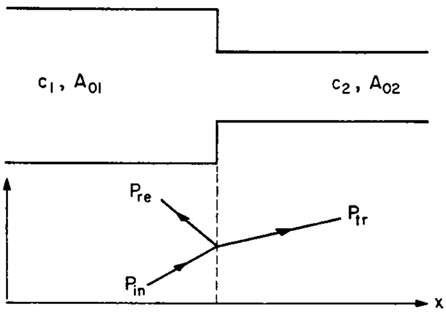

FIGURE 1 A step junction. c_1, c_2, A_{01}, and A_{02} are the elastic wavespeeds and rest areas on both sides of the junction. Arrows show direction of incident, reflected, and transmitted waves on an x-t diagram.

transmitted wave, P_{tr} are set up. It has been shown (3) that in this case

$$P_{re} = \Gamma P_{in} \tag{4}$$

$$P_{tr} = (1 + \Gamma) P_{in} \tag{5}$$

where

$$\Gamma = \frac{1 - (c_1/A_{01})/(c_2/A_{02})}{1 + (c_1/A_{01})/(c_2/A_{02})}. \tag{6}$$

A_{01} and c_1 are the rest area and elastic wavespeed of the upstream tube and A_{02} and c_2 refer to the downstream tube. $c_1 \equiv [(A/\rho)\cdot(dp/dA)]^{1/2}$, evaluated for tube "1" at $A = A_0$.

Consider now a tube with continuously varying properties. For such a tube, a quantity $c(x)$ may be defined as

$$c(x) \equiv [(A_0(x)/\rho)\cdot(\partial p/\partial A)_x{}^0]^{1/2}$$

where ρ is the fluid density and $(\partial p/\partial A)_x^0$ is defined as $\partial p(A, x)/\partial A$ at constant x, evaluated at $A = A_0$. $(\partial p/\partial A)_x^0$, which is referred to as $E(x)$, is a measure of the local tube extensibility.

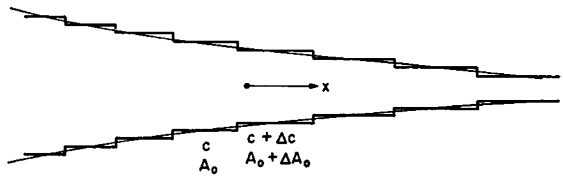

FIGURE 2 Representation of a tube with continuously varying properties by a series of discrete step junctions.

A tube with continuously varying properties may be approximated by a series of very short, lengthwise invariant tubes, each with slightly different properties as in Fig. 2. Consider a step wave of amplitude $P(x_i)$ incident upon the ith junction. Proximal to the junction, $c/A = c(x_i)/A_0(x_i)$ and distal to the junction, $c/A = c(x_i)/A_0(x_i) + \Delta(c/A_0)$ where $\Delta(c/A_0)$ is the change of c/A_0 across the junction. Putting these expressions for c_1/A_{01} and c_2/A_{02} into equation 6 and multiplying both numerator and denominator by $c/A_0 + \Delta(c/A_0)$ we get

$$\Gamma_i = \left[\frac{\Delta(c/A_0)}{2(c/A_0) + \Delta(c/A_0)}\right]_{x_i}. \tag{7}$$

Equation 5 may be rewritten as

$$P_{tr} - P_{in} = \Gamma P_{in} \quad \text{or} \quad \Delta P(x_i) = \Gamma_i\, P(x_i). \tag{8}$$

Putting equation 7 into equation 8, dividing both sides by $P(x_i)$, and taking the limit as the number of junctions become infinite and $\Delta(c/A_0)$ and ΔP at each junction become infinitesimal, we have, that over a distance, dx

$$\frac{dP}{P} = \frac{d(c/A_0)}{2(c/A_0)} \tag{9}$$

where P is the amplitude of the forward running step when it reaches the position x.

Integrating equation 9 from $x = 0$ to $x = x$ yields the desired result,

$$P(x)/P(0) = \{[c(x)/A_0(x)]/[c(0)/A_0(0)]\}^{1/2}. \tag{10}$$

Since equation 9 shows that the reflected waves are infinitesimal with respect to the incident wave, the forward velocity of propagation of the wave front is, in the limit, equal to the local characteristic propagation velocity, namely, $c(x)$. Also, $V(x)$ must simply be $P(x)/\rho c(x)$.

Thus we see that as a wave front propagates along a section of elastic tube where the local impedance, c/A_0, is changing with distance, a continuous stream of infinitesimal reflections is generated and the integrated effect of these reflections is to modify the wave front in accordance with equation 10.

A similar approach to this problem may be found in reference 14.

BEHAVIOR OF SINUSOIDAL WAVES: GENERAL REMARKS

Since the pressures and flows of the cardiovascular system are observed to be relatively periodic over several cycles and the nonlinear effects have been shown to be small, we can concern ourselves only with sinusoidal disturbances without any loss of generality. For sinusoidal disturbances, equations 1–3 are easily solved numerically.

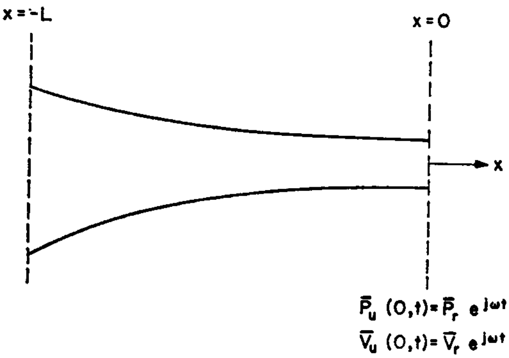

FIGURE 3 Diagrammatic representation of a tube with lengthwise varying properties, showing boundary conditions.

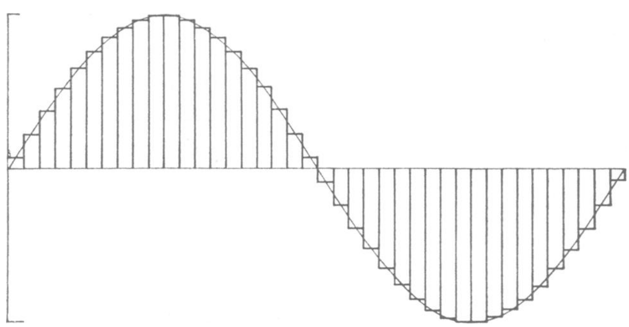

FIGURE 4 Representation of a sinusoidal wave by means of a series of narrow rectangular pulses.

Because $p(A, x)$ does not depend explicitly on time, we may use Taylor's theorem to expand it about the rest condition, $p(A_0, x)$. Retaining only first order terms we have

$$p(A, x) = p(A_0, x) + (\partial p/\partial A)_x^0 (A - A_0). \tag{11}$$

But $p(A_0, x) \equiv p_0$ and $A - A_0 = A_u$ so that equating the expressions for $p(A, x)$

from equations 3 and 11 reveals that $P_u = (\partial p/\partial A)_x^0 A_u$ or $A_u = P_u/E$. Putting this result into equation 2 we have

$$E\partial(A_0 V_u)/\partial x + \partial P_u/\partial t = 0. \tag{12}$$

We make, in equations 1 and 12, the substitution, $\bar{P}_u(x, t) = [P_c(x) - jP_s(x)] \cdot \exp j\omega t$ and $\bar{V}_u(x, t) = [V_c(x) - jV_s(x)] \exp j\omega t$ where $j = (-1)^{1/2}$ and ω is the angular frequency of the sinusoidal disturbance. Separating the equations into real and imaginary parts yields four simultaneous ordinary differential equations for P_c, P_s, V_c, and V_s. As a boundary value problem, these equations were easily solved on an IBM 360 computer (International Business Machines Corp., New York), using the Runge-Kutta technique. The appropriate boundary conditions are shown in Fig. 3. The real solutions and boundary conditions are simply the real part of the complex quantities. (Complex numbers are denoted by a bar atop them.)

A typical solution is shown in Fig. 5. a_p and a_v are the apparent phase velocities of the sinusoidal pressure and velocity waves and the subscript "r" refers to conditions at $x = 0$. Two limiting types of behavior are immediately seen; one for ω approaching zero and another for ω approaching infinity. Plotting the results in dimensionless form reveals an interesting result. It is found that for any numerical value of l, the solutions for $|\bar{P}_u/\bar{P}_r|$, $|\bar{V}_u/\bar{V}_r|$, a_p/c_r, and a_v/c_r closely follow the zero frequency limit over the interval $x = 0$ to $x = -l$ whenever $\omega l/c_r < 0.5$, whereas whenever $\omega l/c_r > 5$, they have an average behavior, over that interval, like the infinite frequency limit. Since this is the case, we make the interesting observations that for any frequency the exact behavior over lengths less than 0.5 c_r/ω (roughly one-tenth of a wavelength) is described by the zero frequency limit whereas over lengths greater than a wavelength (5.0 c_r/ω), the *average* behavior is described by the infinite frequency limit. The exact behavior over a distance less than one-tenth of a wavelength may be quite different from the average behavior over many wavelengths, as reference to Fig. 5 shows. In Fig. 6, data for a single frequency are plotted vs. x/λ_r (where $\lambda_r \equiv 2\pi c_r/\omega$) in order to further illustrate this point.

BEHAVIOR OF SINUSOIDAL WAVES: PHYSICAL ARGUMENTS

A sinusoidal wave may be thought of as composed of a series of narrow rectangular pulses (Fig. 4). In describing the forward propagation of a sinusoidal wave (in a linear system), we need only describe the sum total behavior of those rectangular pulses which are propagating forward and the infinite number of infinitesimal rectangular pulses which are reflected backwards from the forward propagating ones. In addition, as each of the reflected waves propagates backward, secondary, and in fact tertiary and higher order reflections are also generated. Each narrow rectangular pulse is simply a positive wave front followed an instant later by a negative wave front and, as such, behaves as previously discussed.

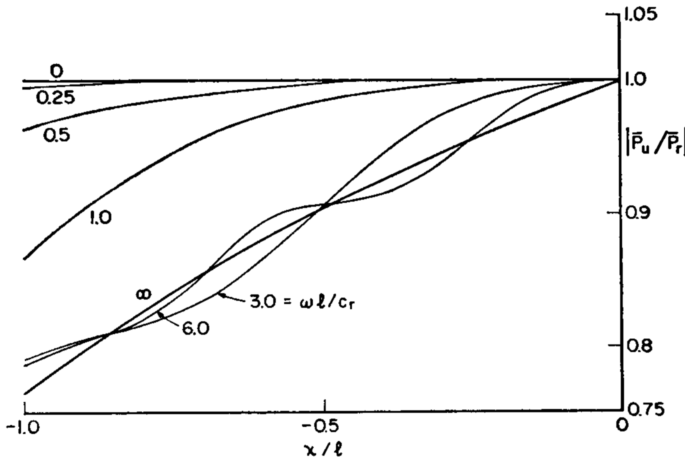

FIGURE 5*a*

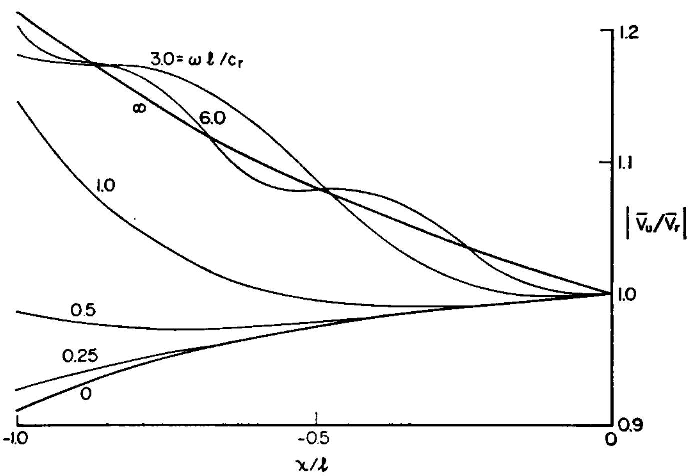

FIGURE 5*b*

FIGURE 5 Results of the computer solutions for a tube where $A_0/A_r = 1.0$, $E/E_r = 1.0 + 0.6(x/l)$ for x/l between 0 and -1 and $Q_m/A_rc_r = 0.063$, $\bar{P}_r/\rho c_r \bar{V}_r = 1.0$, where Q_m is the mean volume flow, assumed zero elsewhere in the paper. Figs. *a*, *b*, *c*, and *d* show the relative pressure and velocity amplitudes, and the apparent phase velocities of both pressure and velocity waves as a function of distance along the tube. Note: (1) the existence of a zero frequency limit, (2) the existence of an infinite frequency limit, (3) the oscillation of the solutions for high $\omega l/c_r$ about the infinite frequency limit, and (4) that a net flow where Q_m/A_rc_r is small does not radically change the behavior discussed in the text. (For the case of nonzero mean flow, equations 1 and 12 are somewhat modified.) The solutions displayed in this manner are independent of the choice of l.

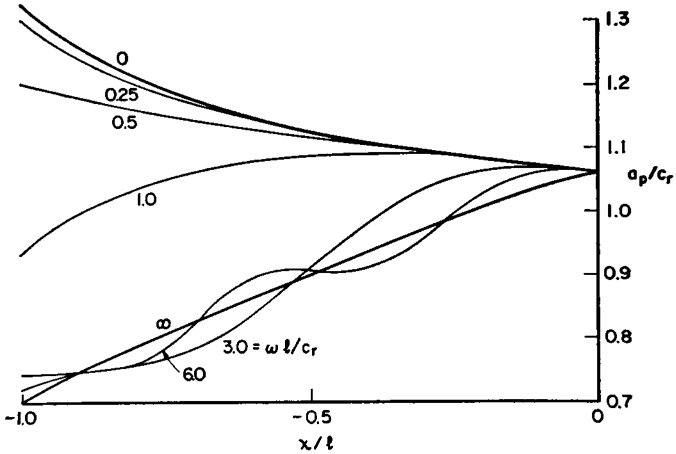

FIGURE 5c

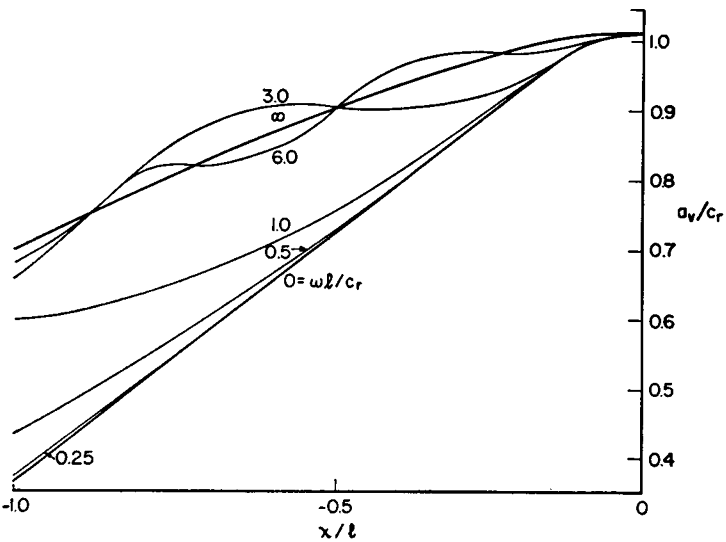

FIGURE 5d

In calculating the net reflected wave, we consider, as a special case, an elastic tube where c is constant and imagine a sinusoidal wave (composed of all its rectangular pulses) travelling forward and to the right in a nonuniform tube such as in Fig. 3. We have argued that one description of such a wave would be

$$P_f(x) = P_r[(c/A_0)/(c_r/A_r)]^{1/2} \cos(\omega t - \omega x/c). \tag{13}$$

As the wave propagates forward, there are back reflections set up at each position. A reflection generated at the point, X, at time T will reach the point x_1, at

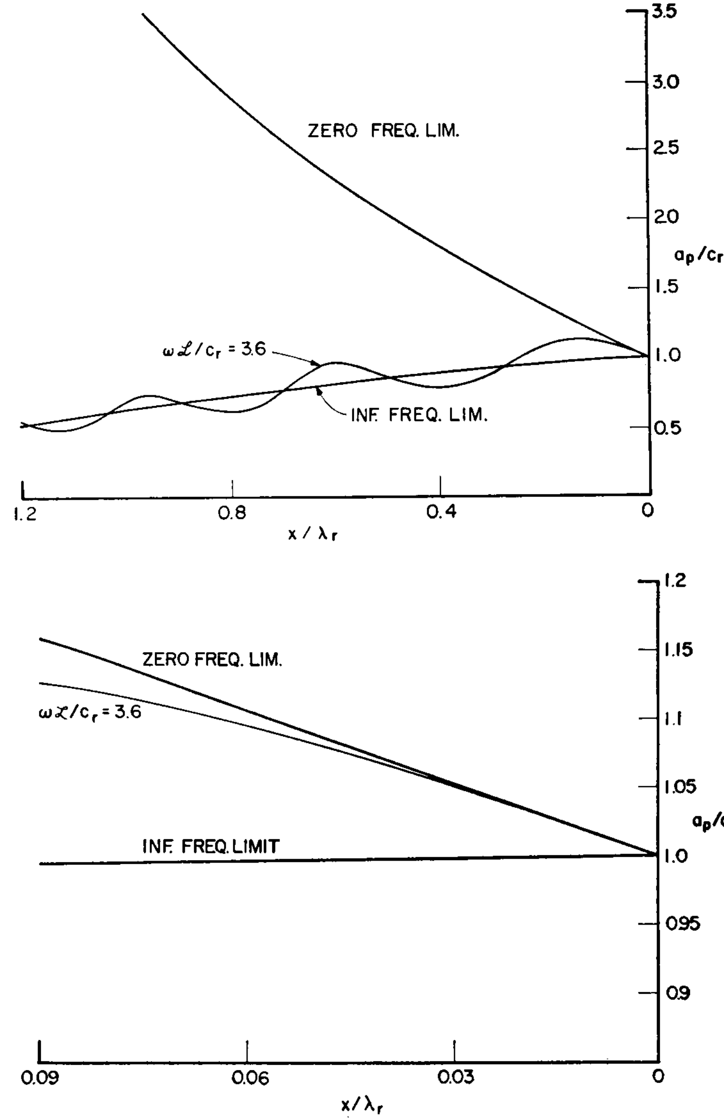

FIGURE 6 a_p/c_r vs. x/λ_r for the frequency $\omega = 200$ rad/sec in a tube where $A_0/A_r = 1.0 - 2.52(x/36) + 0.648(x/36)^2 - 1.87(x/36)^3$; $E/E_r = 1.0 + 2.5356(x/36) + 2.3063(x/36)^2 + 0.7347(x/36)^3$ for x between 0 and -36, and where $Q_m/A_r c_r = 0$, and $\overline{P}_r/\rho c_r \overline{V}_r = 1.0$. The area around $x/\lambda_r = 0$ is expanded in Fig. 6*b*. Note that although over many wavelengths the exact solution oscillates about the infinite frequency limit, over a fraction of a wavelength it follows the zero frequency limit.

time, t_1 if and only if

$$X - x_1 = c(t_1 - T). \tag{14}$$

The amplitude of the forward wave at X at time, T, is, from equation 13

$P_r[(c/A_0)/(c_r/A_r)]_X^{1/2}\cos(\omega T - \omega X/c)$ so that the amplitude of the reflected wave there, from equation 9, is

$$dP = P_r[(c/A_0)/(c_r/A_r)]_X^{1/2}\cos(\omega T - \omega X/c)\left[(2c/A_0)^{-1}\frac{d(c/A_0)}{dx}\right]_X dX. \quad (15)$$

Since we are interested only in the waves which originate from X at time T and arrive at x_1 at time, t_1, we have $T = t_1 - (X - x_1)/c$. Putting this into equation 15, and bearing in mind that because of second order reflections the primary reflection will have its amplitude modified by the factor $[(c/A_0)_{x_1}/(c/A_0)_X]^{1/2}$ in travelling from X to x_1, we find, by summing over all X, the net sum at point x_1 and time, t_1, of all the first order reflections generated by the right-running sinusoid, namely

$$\sum \text{all reflections} = P_{br} = \int_{X=x_1}^{X=0} dP$$

$$= \int_{X=x_1}^{X=0} P_r[(c/A_0)_{x_1}/(c_r/A_r)]^{1/2}[\cos(\omega t_1 + \omega x_1/c - 2\omega X/c)]$$

$$\cdot\left[(2c/A_0)^{-1}\frac{d(c/A_0)}{dx}\right]_X dX. \quad (16)$$

It should be noted that although the effect of second order reflections on the primary reflection must be taken into account, the second order reflections themselves (and also higher order reflections) do not contribute to the final pressure distribution. This is because, from equations 9 and 15, they are of order $(dP)^2$ or less.

For ease of further calculation, we restrict ourselves to the case where the tube we are discussing is impedance-matched at its distal end so that no gross reflections arise from its termination. In this case the total disturbance is simply P_f from equation 13 plus P_{br} from equation 16. Dropping the subscript "1" and expanding the cosine function in equation 16, we have that

$$P_{br} = P_r[(c/A_0)_x/(c_r/A_r)]^{1/2}[S_1(x)\cos(\omega t + \omega x/c) + S_2(x)\sin(\omega t + \omega x/c)] \quad (17)$$

where
$$S_1 = (-\tfrac{1}{2})\int_0^x \left[(c/A_0)^{-1}\frac{d(c/A_0)}{dx}\right]_{x=X}\cos(2\omega X/c)\,dX,$$

$$S_2 = (-\tfrac{1}{2})\int_0^x \left[(c/A_0)^{-1}\frac{d(c/A_0)}{dx}\right]_{x=X}\sin(2\omega X/c)\,dX.$$

The sum of P_f and P_{br} may now easily be put in amplitude-phase form and the resulting amplitude is

$$|P_u/P_r| = [(c/A_0)/(c_r/A_r)]^{1/2}[1 + S_1^2 + S_2^2$$

$$+ 2S_1\cos(2\omega x/c) + 2S_2\sin(2\omega x/c)]^{1/2} \quad (18)$$

One may, by repeated application of integration by parts, show that if all the derivatives of $f(x)$ are finite, then in the limit of large ω/c

$$\int_0^x f(x) \cos (2\omega x/c)\, dx \sim (c/2\omega) f(x) \sin (2\omega x/c), \tag{19}$$

$$\int_0^x f(x) \sin (2\omega x/c)\, dx \sim (-c/2\omega)[f(x) \cos (2\omega x/c) - f(0)]. \tag{20}$$

Using equations 19 and 20 to calculate S_1 and S_2 for the case of large ω/c (or, in nondimensional form, large $\omega l/c$) and placing these expressions into equation 18 while dropping terms of second order in c/ω, we get the result that for large $\omega l/c$

$$| P_u/P_r | = [(c/A_0)/(c_r/A_r)]^{1/2} \left[1 - \frac{1}{\pi} (4\mathcal{L}/\lambda)^{-1}_{x=0} \sin 2\omega x/c \right]^{1/2} \tag{21}$$

where $\mathcal{L}^{-1} = (c/A_0)^{-1}\, d(c/A_0)/dx$.

Although we have restricted ourselves somewhat in the above calculations by assuming $c(x)$ constant, we may conclude, in general, that:

(*a*) In the limit of infinite frequency, the net effect of all the back reflections is zero and a sinusoidal wave behaves just as a step wave.

(*b*) For large $\omega l/c_r$, that is, over distances of several wavelengths, the exact solution for a sinusoidal wave oscillates about the infinite frequency solution with a wavelength of roughly $(1/2)(2\pi c(x)/\omega)$, and therefore the infinite frequency (high $\omega l/c_r$) limit represents the wave's average behavior over several wavelengths.

(*c*) The percentage amplitude of the oscillation depends upon $4\mathcal{L}/\lambda$ and for $\mathcal{L}/\lambda > 1$ is of order $(4\mathcal{L}/\lambda)^{-1}/2\pi$. $4\mathcal{L}/\lambda$ is a measure of how much c/A_0 changes over a distance of one quarter wavelength. The smaller the change of c/A_0 per quarter wavelength, the larger $4\mathcal{L}/\lambda$ is.

(*d*) These oscillations are due to the summation of the back reflected waves with the forward moving waves.

(*e*) Because of these reflections, the exact behavior of a sinusoidal wave over a fraction of a wavelength can vary considerably from its average behavior and the behavior of a step wave.

ANALYTIC EXPRESSIONS FOR THE ZERO AND INFINITE FREQUENCY LIMITS

High $\omega l/c_r$

We have shown that, in a system where the nonlinearities are not large, in the limit of infinite frequency each part of a sinusoidal wave behaves just as the front of a step wave. That is, its pressure amplitude varies as $[(c/A_0)/(c_r/A_r)]^{1/2}$ and each wave element travels with the local speed, c. $P(x) \exp j(\omega t + \int_0^x (-\omega/c)\, dx)$ is the equation of a sinusoidal wave whose amplitude is $P(x)$ and whose apparent phase velocity is $c(x)$. This simply follows from the relationships for sinusoidal waves that (p. 217, reference 5)

$$a_p = -\omega(d\theta_p/dx)^{-1}; \qquad a_v = -\omega(d\theta_v/dx)^{-1} \tag{22}$$

where θ_p and θ_v are the respective phases of the pressure and velocity waves. Therefore, when, in general, we have both right-running and left-running waves, the solutions for $\bar{P}_u$ and $\bar{V}_u$, as ω approaches infinity are

$$\bar{P}_u(x, t) = \left[\frac{\bar{P}_r}{1 + \bar{\Gamma}}\right][(c/A_0)/(c_r/A_r)]^{1/2}[e^{j(\omega t+\phi)} + \bar{\Gamma}e^{j(\omega t-\phi)}] \tag{23}$$

$$\bar{V}_u(x, t) = \left[\frac{\bar{P}_r}{(1 + \bar{\Gamma})\rho c(x)}\right][(c/A_0)/(c_r/A_r)]^{1/2}[e^{j(\omega t+\phi)} - \bar{\Gamma}e^{j(\omega t-\phi)}] \tag{24}$$

where

$$\phi(x) \equiv \int_0^x [-\omega/c(x)]\, dx \tag{25}$$

and where $\bar{\Gamma}$ is the complex ratio of the left-running to right-running waves at $x = 0$ (often called the reflection coefficient at $x = 0$). In line with the linear approximations we have made, $\bar{\Gamma}$ may be found from the boundary conditions by means of the relationship (pp. 23–25, reference 13)

$$\frac{\bar{P}_r}{\bar{V}_r} = \rho c_r \frac{1 + \bar{\Gamma}}{1 - \bar{\Gamma}}. \tag{26}$$

In a vessel where c/A_0 is constant along the entire length, equations 23 and 24 are the exact solutions for all frequencies (i.e., they satisfy equations 1 and 12). One might suspect this, as this is the case in which there are no back reflections generated. The uniform tube, c and A_0 both constant, is simply a special case of c/A_0 constant.

Low $\omega l/c_r$

An analytic expression for the limit of low $\omega l/c_r$ may be found by means of a perturbation analysis which is valid whenever $\omega l/c_r$ is sufficiently small. Making the substitution $\bar{P}_u(x, t) = \bar{P}_u^*(x)e^{j\omega t}$ and $\bar{V}_u(x, t) = \bar{V}_u^*(x)e^{j\omega t}$ in equations 1 and 12 and writing the equations in nondimensional form we get

$$E'\, d(A_0'\, \bar{V}_u')/dx' + j(\omega l/c_r)\bar{P}_u' = 0 \tag{27}$$

$$d\bar{P}_u'/dx' + j(\omega l/c_r)\bar{V}_u' = 0 \tag{28}$$

where $\bar{P}_u' \equiv \bar{P}_u^*/\rho c_r^2$, $\bar{V}_u' \equiv V_u^*/c_r$, $c' \equiv c/c_r$, $E' \equiv E/E_r$, $x' \equiv x/l$, $t' \equiv \omega t$, and $A_0' \equiv A_0/A_r$.

We look for solutions of the form

$$\bar{P}_u' = \bar{P}_{00}' + (\omega l/c_r)\bar{P}_{10}' + (\omega l/c_r)^2\bar{P}_{20}' + \cdots \tag{29}$$

$$\bar{V}_u' = \bar{V}_{00}' + (\omega l/c_r)\bar{V}_{10}' + (\omega l/c_r)^2\bar{V}_{20}' + \cdots \tag{30}$$

Putting equations 29 and 30 into equations 27 and 28 and retaining terms of second order or less in $\omega l/c_r$ we have

$$[E'\ d(A_0'\bar{V}_{00}')/dx'] + (\omega l/c_r)[d(A_0'\bar{V}_{10}')/dx' + j\bar{P}_{00}'] + (\omega l/c_r)^2[E'\ d(A_0'\bar{V}_{20}')/dx' + j\bar{P}_{10}'] = 0 \tag{31}$$

$$[d\bar{P}_{00}'/dx'] + (\omega l/c_r)[d\bar{P}_{10}'/dx' + j\bar{V}_{00}'] + (\omega l/c_r)^2[d\bar{P}_{20}'/dx' + j\bar{V}_{10}'] = 0. \tag{32}$$

We have as boundary conditions, $\bar{P}_u'(0) = \bar{P}_r/\rho c_r^2$ and $\bar{V}_u'(0) = \bar{V}_r/c_r$, or equivalently, at $x = 0$; $\bar{P}_{00}' = \bar{P}_r/\rho c_r^2$, $\bar{V}_{00}' = \bar{V}_r/c_r$, and $\bar{P}_{10}' = \bar{P}_{20}' = \bar{V}_{10}' = \bar{V}_{20}' = 0$.

In order for equations 31 and 32 to be satisfied for all small values of $\omega l/c_r$, each of the terms enclosed in brackets must be identically zero. Using the boundary conditions, this enables us to solve directly for $\bar{P}_{00}'$, $\bar{P}_{10}'$, $\bar{V}_{00}'$, etc.

The full solution in dimensional form (out to second order in $\omega l/c_r$) is

$$\bar{P}_u(x,t) = \bar{P}_r\left\{1 - \beta j[\rho c_r\ \bar{V}_r/\bar{P}_r] \int_0^x [A_r/A_0]\ d[x/l] - \beta^2 \int_0^x [A_r/A_0]\left[\int_0^x [E_r/E]\ d[x/l]\right] d[x/l]\right\} e^{j\omega t} \tag{33}$$

$$A_0(x)\ \bar{V}_u(x,t) = A_r\ \bar{V}_r\left\{1 - \beta j[\bar{P}_r/\rho c_r\bar{V}_r] \int_0^x [E_r/E]\ d[x/l] - \beta^2 \int_0^x [E_r/E]\left[\int_0^x [A_r/A_0]\ d[x/l]\right] d[x/l]\right\} e^{j\omega t} \tag{34}$$

where $\beta \equiv \omega l/c_r$.

RANGE OF VALIDITY OF THE ANALYTIC EXPRESSIONS

The only assumption made in deriving the low $\omega l/c_r$ solutions were that v/c and $\omega l/c$ were small with respect to unity. Therefore, for a given frequency, ω, the solutions are valid over any length, l, so long as over the entire interval, v/c and $\omega l/c$ are small. This is independent of the magnitude of the total change in A_0, c, or E. In our computer runs we found the computer solutions to agree well with equations 33 and 34, out to first order, whenever $\omega l/c_r$ was less than 0.5. Comparison with the solutions out to second order was not made.

The appropriateness of using the infinite frequency solutions in describing the behavior of a finite frequency depends upon how greatly the solutions for the finite frequency oscillate about the infinite frequency solutions. Fig. 7 shows a very crude estimate of how large a value of $\omega l/c_r$ is required for different values of $\mathcal{L}/\lambda_r$ so that the maximum amplitude of oscillation over the interval, l, is less than a certain per cent of the pressure amplitude at $x = -l$. This has been estimated for the

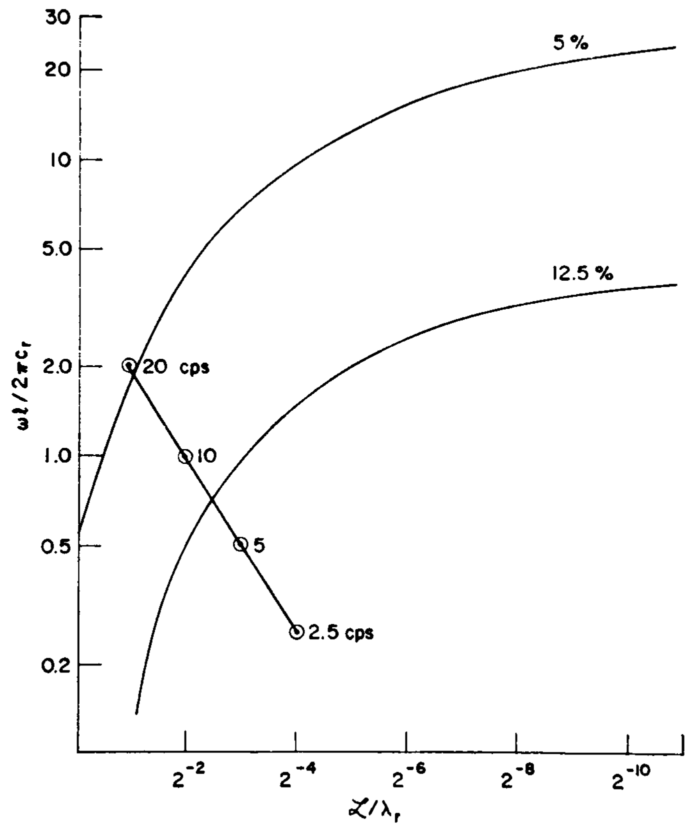

FIGURE 7 An estimate of how large a value of $\omega l/c_r$ is required for a given value of $\mathcal{L}/\lambda_r$ so that the maximum amplitude of oscillation over the interval, l, is less than 5 and 12.5% of the pressure amplitude at $x = -l$. See text. Circles indicate the approximate position of certain frequencies in such a plot for the dog's aorta, based on currently accepted data (references 10, 11).

particular case where c/A_0 increases linearly from an impedance-matched end at $x = 0$. The approximate theoretical position on the graph of several frequencies for the dog aorta is also indicated.

The use of the expression "infinite frequency solution" is somewhat of a misnomer since as the frequency approaches infinity the assumption that the wavelength of the disturbance in area be large with respect to the diameter of the vessel eventually breaks down. However, in a wide variety of problems, including that of pulse wave propagation in the cardiovascular system, the higher frequencies of interest are not so high as to violate any of the assumptions of the present model, but yet are high enough so that over the distances of interest their average behavior is described by the high $\omega l/c_r$ solutions.

DISCUSSION

We have seen that the behavior of sinusoidal waves in vessels where the area and distensibility vary with distance may be explained, to a large extent, by considering the effect of the continuous stream of infinitesimal reflections that is set up whenever a wave travels in a region of vessel where the local impedance, c/A_0, is not constant.

For a vessel of fixed length, L, we may make the following conclusions: Those frequencies for which $\omega L/c_r$ is less than 0.5, that is, those for which the tube is but a fraction of a wavelength, behave, along the length of the tube, as described by equations 33 and 34. Those for which $\omega L/c_r$ is greater than 5 have an *average* behavior along the tube as described by equations 23 and 24. However, the *exact* description of these same frequencies over distances, Δx, where $\omega \Delta x/c_r$ is small, is different from its average behavior and the more c/A_0 changes over this interval the greater the difference. In fact, over a distance of the tube, Δx, where $\omega \Delta x/c_r$ is less than 0.5, the behavior of the wave is described by the low $\omega l/c_r$ solutions. The exact solution over many wavelengths, we have seen, oscillates about the average solution with a wavelength of roughly $(\frac{1}{2})(2\pi c(x)/\omega)$.

TABLE I

	$\omega l/c_r \rightarrow 0$	$\omega l/c_r \rightarrow \infty$
$\lvert \bar{P}_u/\bar{P}_r \rvert$	1	$[(c/A_0)/(c_r/A_r)]^{1/2}$
$\lvert \bar{V}_u/\bar{V}_r \rvert$	A_r/A_0	$(c_r/c)[(c/A_0)/(c_r/A_r)]^{1/2}$
a_p/c_r	A_0/A_r	c/c_r
a_v/c_r	E/E_r	c/c_r

The relative pressure and velocity magnitudes and the apparent phase velocities of a sinusoidal wave travelling in a nonuniform terminally impedance-matched elastic tube, found from equations 23, 24, 33, and 34. Second order and higher terms in $\omega l/c_r$ have been dropped.

Equations 23, 24, 33, and 34 may be conveniently summarized by using them to find $\lvert \bar{P}_u/\bar{P}_r \rvert$, $\lvert \bar{V}_u/\bar{V}_r \rvert$, a_p/c_r, and a_v/c_r. This is easily done by first putting them into amplitude-phase form and then using equation 22 to find the apparent phase velocities. The results for a tube with an impedance-matched end at $x = 0$, that is, one whose termination is such that no gross reflections are set up at it, (i.e. $\bar{\Gamma} = 0$) are summarized in Table I. As in the case of the uniform tube, when the terminal end of the tube is not impedance-matched, the tendency is for the pressure amplitude to increase and the velocity amplitude to decrease near a "closed" type end and vice versa near an "open" type end.

Comparison of Table I or equations 23 and 24 with the results for a uniform tube (equations 23 and 24 with c and A_0 constant) shows that the *average behavior over several wavelengths* of a sinusoidal wave in a tube with lengthwise varying properties is not too dissimilar from the behavior of waves in a lengthwise invariant tube, the differences being that the constant wavespeed, $\pm c_r$, for incident and reflected waves is replaced by $\pm c(x)$ and furthermore, that both incident and reflected waves have their amplitudes modified by the factor $[(c/A_0)/(c_r/A_r)]^{1/2}$. On the other hand, examination of equations 33 and 34 or Table I shows that over a distance which is *a fraction of a wavelength*, the behavior in a nonuniform tube is quite unlike that in a uniform tube, particularly with regard to the apparent phase velocities.

Further Conclusions of Physiological Interest

In the dog aorta, where c_r is roughly 500 cm/sec and the length approximately 36 cm, all frequencies above 3 cps (corresponding to $\omega l/c_r \sim 1.4$) begin to have an average behavior over the length of the aorta like that of very high frequencies. In particular, the pressure amplitude, disregarding reflections from junctions and terminal beds, increases in the direction of increasing $(c/A_0)^{1/2}$ and the velocity amplitude, under the same conditions goes as $c^{-1}(c/A_0)^{1/2}$. It follows that the physiologically observed increase of the pressure pulse in the large and medium size arteries of the dog, where the effects of viscosity are relatively unimportant, may be explained by the increase in these vessels of $(c/A_0)^{1/2}$ or, by the positive type reflections which have been shown (5) to exist there.

Both the present work and that of Barnard et al. (1) have shown that when the area is constant and elastic wavespeed increases with distance, then the velocity pulse decays with distance. However, in a vessel such as the dog aorta, the increase in c along the vessel is more than balanced by the decrease in A_0 so that $c^{-1}(c/A_0)^{1/2}$ increases with distance. Therefore, the observed decrease in the velocity pulse in this vessel is not due so much to changes in tube properties but presumably to both viscosity and closed (positive) type reflections. Womersley (18) has analyzed both the iliac junction and aortic trifurcation of the dog and found them to cause positive type reflections. To what extent the human aorta is similar to the dog's remains to be further delineated, but we may guess that the effects are at least qualitatively the same.

It has been popular to study vessel properties, particularly $(\partial p/\partial A)_x$, by measuring either the apparent phase velocity of a particular harmonic or the velocity of propagation of the foot of the pressure pulse under conditions where reflections are of minimal importance. The present study indicates that in a vessel like the aorta, where c/A_0 may change by a factor of 12 over its length, one must be careful to measure the apparent phase velocity of a frequency whose wavelength is at least comparable to, if not less than, the length over which one wishes to discern changes in $(\partial p/\partial A)_x$. Even then, the present study indicates that with refined measuring techniques the apparent phase velocity as a function of x will be found to oscillate about and not exactly follow $c(x)$. The wavelength of this oscillation is roughly half that that would be caused by the presence of reflecting sites and standing waves.

Oscillations should also be present in a plot of pressure or velocity amplitude (for a single harmonic) vs. distance along the vessel. This effect would be easier to detect since amplitude may be measured at a point whereas the measured apparent phase velocity must be the average velocity between two points.

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